# Overview of Complex Networks

## Complex Networks, SFI Summer School, June, 2010

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# Outline



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Basic definitions
Popularity
Examples of Complex Networks
Properties of Complex Networks
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# Something of a plan:

- Lecture 1: Overview; Background
- Lecture 2: Random, Scale-free, and Small-World networks
- Lecture 3: Models of Contagion
- Lecture 4: Transportation networks; Discovering structure

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# Exciting details regarding these slides:

- Three versions (all in pdf):
  1. Presentation,
  2. Flat Presentation,
  3. Handout (2x2).
- Presentation versions are navigable and hyperlinks are clickable.
- Web links look like this (⊞).
- References in slides link to full citation at end. [2]
- Citations contain links to papers in pdf (if available).
- 50 hours of lectures → 5 hours.
- Brought to you by a concoction of LaTeX, Beamer, perl, and madness.

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# Bonus materials:

## Graduate Course Websites:

- SFI Summer School Course (this one!):
  http://www.uvm.edu/∼pdodds/teaching/courses/2010-06SFI-networks/ (⊞)
- Principles of Complex Systems (⊞), University of Vermont
- Complex Networks (⊞), University of Vermont

## Textbooks:

- Mark Newman (Physics, Michigan)
  "Networks: An Introduction" (⊞)
- David Easley and Jon Kleinberg (Economics and Computer Science, Cornell)
  "Networks, Crowds, and Markets: Reasoning About a Highly Connected World" (⊞)





# Bonus materials:

## Review articles:

- S. Boccaletti et al.
  "Complex networks: structure and dynamics" [4]
  Times cited: 1,028 (as of June 7, 2010)
- M. Newman
  "The structure and function of complex networks" [15]
  Times cited: 2,559 (as of June 7, 2010)
- R. Albert and A.-L. Barabási
  "Statistical mechanics of complex networks" [1]
  Times cited: 3,995 (as of June 7, 2010)

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# Basic definitions

## Complex System—Some ingredients:

- Distributed system of many interrelated parts
- No centralized control
- Nonlinear relationships
- Existence of feedback loops
- Complex systems are open (out of equilibrium)
- Presence of Memory
- Modular (nested)/multiscale structure
- Opaque boundaries
- Emergence—'More is Different' [2]

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# Basic definitions

Complex: (Latin = with + fold/weave (com + plex))

Adjective

- Made up of multiple parts; intricate or detailed.
- Not simple or straightforward.

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## Thesaurus deliciousness:

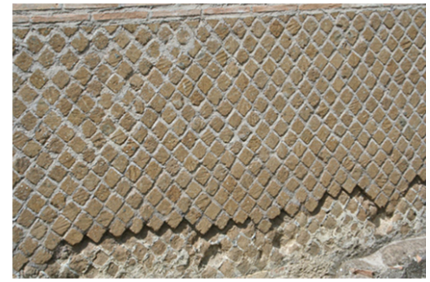
**network**
noun
**1** *a network of arteries* WEB, lattice, net, matrix, mesh, crisscross, grid, reticulum, reticulation; Anatomy plexus.
**2** *a network of lanes* MAZE, labyrinth, warren, tangle.
**3** *a network of friends* SYSTEM, complex, nexus, web, webwork.



## Ancestry:

From Keith Briggs's excellent etymological investigation: (⊞)

- Opus reticulatum:
- A Latin origin?



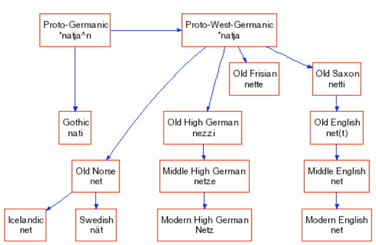
[http://serialconsign.com/2007/11/we-put-net-network]



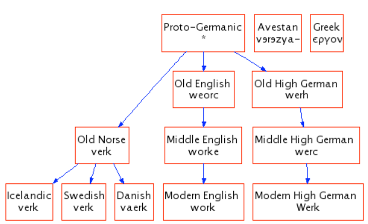
## Ancestry:

### Net and Work are venerable old words:

- 'Net' first used to mean spider web (King Ælfréd, 888).
- 'Work' appears to have long meant purposeful action.




The network of Germanic 'net' words




The network of 'work' words

- 'Network' = something built based on the idea of natural, flexible lattice or web.
- c.f., ironwork, stonework, fretwork.



## Ancestry:

### First known use: Geneva Bible, 1560

'And thou shalt make unto it a grate like networke of brass (Exodus xxvii 4).'

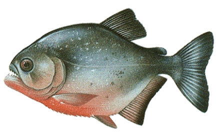
### From the OED via Briggs:

- 1658–: reticulate structures in animals
- 1839–: rivers and canals
- 1869–: railways
- 1883–: distribution network of electrical cables
- 1914–: wireless broadcasting networks

- Natural → man-made
- Physical connections → Wire-less connections → abstract connections

# Key Observation:

- Many complex systems can be viewed as complex networks of physical or abstract interactions.
- Opens door to mathematical and numerical analysis.
- Dominant approach of last decade of a theoretical-physics/stat-mechish flavor.
- Mindboggling amount of work published on complex networks since 1998...
- ... largely due to your typical theoretical physicist:



- *Piranha physicus*
- Hunt in packs.
- Feast on new and interesting ideas (see chaos, cellular automata, ...)





# Popularity (according to ISI Web of Knowledge)

## "Collective dynamics of 'small-world' networks"[21]

- Watts and Strogatz Nature, 1998
- Cited ≈ 4325 times (as of June 7, 2010)
- Over 1100 citations in 2008.

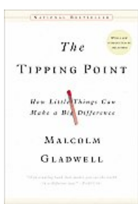
## "Emergence of scaling in random networks"[3]

- Barabási and Albert Science, 1999
- Cited ≈ 4769 times (as of June 7, 2010)
- Over 1100 citations in 2008.

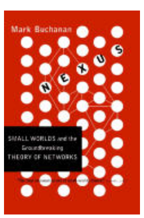




# Popularity according to books:




The Tipping Point: How Little Things can make a Big Difference—Malcolm Gladwell[10]

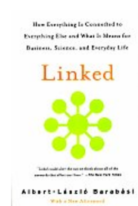



Nexus: Small Worlds and the Groundbreaking Science of Networks—Mark Buchanan

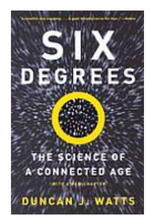




# Popularity according to books:




Linked: How Everything Is Connected to Everything Else and What It Means—Albert-Laszlo Barabási




Six Degrees: The Science of a Connected Age—Duncan Watts[20]

## Numerous others:

- Complex Social Networks—F. Vega-Redondo [19]
- Fractal River Basins: Chance and Self-Organization—I. Rodríguez-Iturbe and A. Rinaldo [16]
- Random Graph Dynamics—R. Durette
- Scale-Free Networks—Guido Caldarelli
- Evolution and Structure of the Internet: A Statistical Physics Approach—Romu Pastor-Satorras and Alessandro Vespignani
- Complex Graphs and Networks—Fan Chung
- Social Network Analysis—Stanley Wasserman and Kathleen Faust
- Handbook of Graphs and Networks—Eds: Stefan Bornholdt and H. G. Schuster [6]
- Evolution of Networks—S. N. Dorogovtsev and J. F. F. Mendes [9]



## More observations

- But surely networks aren't new...
- Graph theory is well established...
- Study of social networks started in the 1930's...
- So why all this 'new' research on networks?
- Answer: Oodles of Easily Accessible Data.
- We can now inform (alas) our theories with a much more measurable reality.*
- A worthy goal: establish mechanistic explanations.

*If this is upsetting, maybe string theory is for you...*

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## More observations

- Web-scale data sets can be overly exciting.

Witness:

- The End of Theory: The Data Deluge Makes the Scientific Theory Obsolete (Anderson, Wired) (⊞)
- "The Unreasonable Effectiveness of Data," Halevy et al. [11].

But:

- For scientists, description is only part of the battle.
- We still need to understand.

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## Super Basic definitions

Nodes = A collection of entities which have properties that are somehow related to each other

- e.g., people, forks in rivers, proteins, webpages, organisms,...

Links = Connections between nodes

- Links may be directed or undirected.
- Links may be binary or weighted.

Other spiffing words: vertices and edges.

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# Super Basic definitions

## Node degree = Number of links per node

- Notation: Node $i$'s degree = $k_i$.
- $k_i = 0, 1, 2, \ldots$.
- Notation: the average degree of a network = $\langle k \rangle$ (and sometimes $z$)
- Connection between number of edges $m$ and average degree:

$$\langle k \rangle = \frac{2m}{N}.$$

- Defn: $\mathcal{N}_i$ = the set of $i$'s $k_i$ neighbors





# Super Basic definitions

## Adjacency matrix:

- We represent a directed network by a matrix $A$ with link weight $a_{ij}$ for nodes $i$ and $j$ in entry $(i, j)$.
- e.g.,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

- (n.b., for numerical work, we always use sparse matrices.)

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# Examples

## So what passes for a complex network?

- Complex networks are large (in node number)
- Complex networks are sparse (low edge to node ratio)
- Complex networks are usually dynamic and evolving
- Complex networks can be social, economic, natural, informational, abstract, ...

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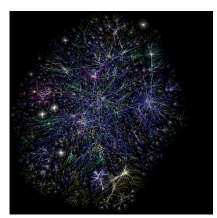
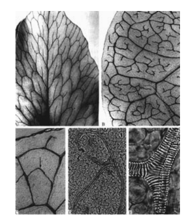
# Examples

## Physical networks

- River networks
- Neural networks
- Trees and leaves
- Blood networks
- The Internet
- Road networks
- Power grids





- Distribution (branching) vs. redistribution (cyclical)

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# Examples

## Interaction networks

- The Blogosphere
- Biochemical networks
- Gene-protein networks
- Food webs: who eats whom
- The World Wide Web (?)
- Airline networks
- Call networks (AT&T)
- The Media
- Paper citations

datamining.typepad.com (⊞)

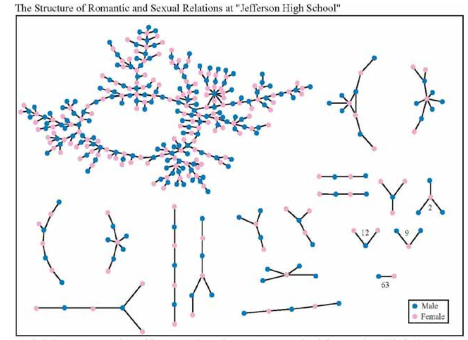


# Examples

## Interaction networks: social networks

- Snogging
- Friendships
- Acquaintances
- Boards and directors
- Organizations
- facebook.com (⊞), twitter.com (⊞)

The Structure of Romantic and Sexual Relations at "Jefferson High School"




Each circle represents a student and lines connecting students represent romantic relations occuring within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (i.e. we found 63 pairs unconnected to anyone else).

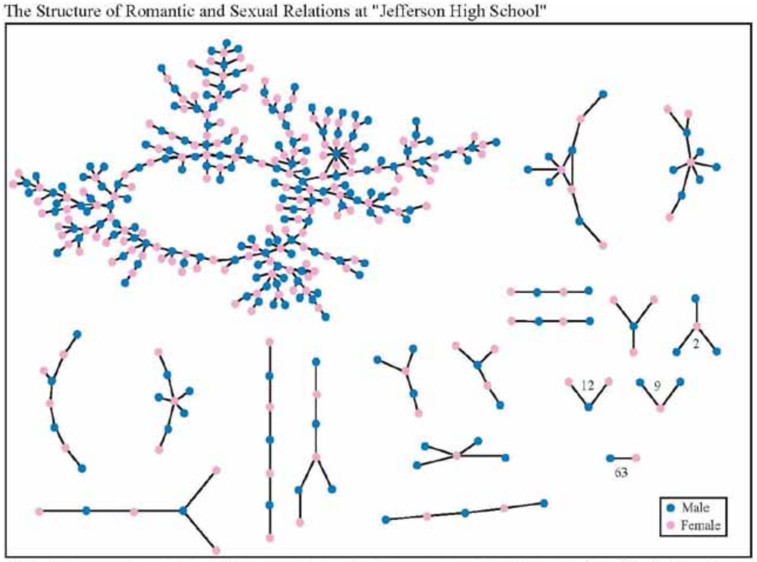
(Bearman *et al.*, 2004)

- 'Remotely sensed' by: email activity, instant messaging, phone logs (*cough*).



# Examples

The Structure of Romantic and Sexual Relations at "Jefferson High School"




Each circle represents a student and lines connecting students represent romantic relations occuring within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (i.e. we found 63 pairs unconnected to anyone else).



# Examples

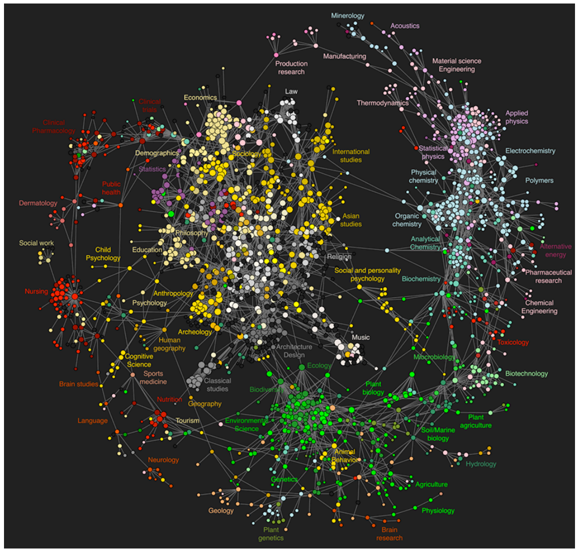
## Relational networks

- Consumer purchases (Wal-Mart: $\approx$ 1 petabyte $= 10^{15}$ bytes)
- Thesauri: Networks of words generated by meanings
- Knowledge/Databases/Ideas
- Metadata—Tagging: del.icio.us (⊞), flickr (⊞)

**common tags** cloud | list

community daily dictionary education encyclopedia english free imported info information internet knowledge learning news reference research resource resources search tools useful web web2.0 wiki wikipedia

## Clickworthy Science:



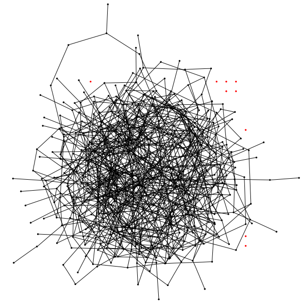
Bollen et al. [5]





## A notable feature of large-scale networks:

- Graphical renderings are often just a big mess.



$\Leftarrow$ Typical hairball
- number of nodes $N = 500$
- number of edges $m = 1000$
- average degree $\langle k \rangle = ?4$

- And even when renderings somehow look good: "That is a very graphic analogy which aids understanding wonderfully while being, strictly speaking, wrong in every possible way" said Ponder [Stibbons] —*Making Money*, T. Pratchett.
- We need to extract digestible, meaningful aspects.





## Properties

### Some key features of real complex networks:

- Degree distribution
- Assortativity
- Homophily
- Clustering
- Motifs
- Modularity
- Concurrency
- Hierarchical scaling
- Network distances
- Centrality
- Efficiency
- Robustness

- Coevolution of network structure and processes on networks.





## Properties

### 1. Degree distribution $P_k$

- $P_k$ is the probability that a randomly selected node has degree $k$
- Big deal: Form of $P_k$ key to network's behavior
- ex 1: Erdős-Rényi random networks have a Poisson distribution:
$$P_k = e^{-\langle k \rangle} \langle k \rangle^k / k!$$
- ex 2: "Scale-free" networks: $P_k \propto k^{-\gamma} \Rightarrow$ 'hubs'
- We'll come back to this business soon...

# Properties



## 2. Assortativity/3. Homophily:

- Social networks: Homophily (⊞) = birds of a feather
- e.g., degree is standard property for sorting: measure degree-degree correlations.
- Assortative network:[14] similar degree nodes connecting to each other.
  - *Often social: company directors, coauthors, actors.*
- Disassortative network: high degree nodes connecting to low degree nodes.
  - *Often techological or biological: Internet, protein interactions, neural networks, food webs.*



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## 4. Clustering:

- Your friends tend to know each other.
- Two measures:

$$C_1 = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i \text{ due to Watts \& Strogatz}^{[21]}$$

$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}} \text{ due to Newman}^{[15]}$$

- $C_1$ is the average fraction of pairs of neighbors who are connected.
- Interpret $C_2$ as probability two of a node's friends know each other.



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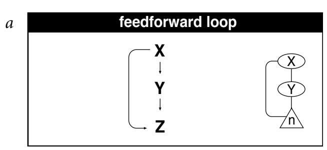
## 5. Motifs:

- Small, recurring functional subnetworks
- e.g., Feed Forward Loop:




Shen-Orr, Uri Alon, *et al.*[17]



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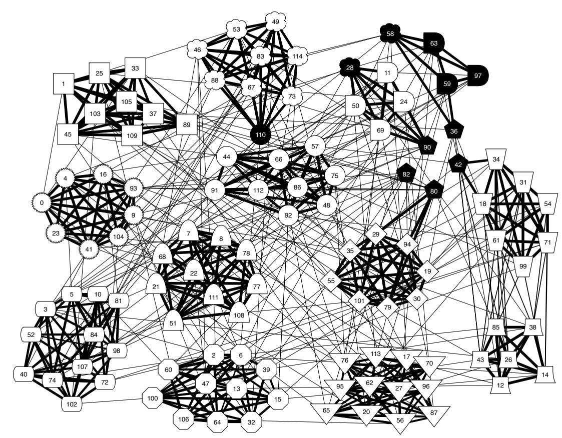
## 6. modularity:



Clauset *et al.*, 2006[7]: NCAA football

# Properties

## 7. Concurrency:

- Transmission of a contagious element only occurs during contact[13]
- Rather obvious but easily missed in a simple model
- Dynamic property—static networks are not enough
- Knowledge of previous contacts crucial
- Beware cumulated network data!

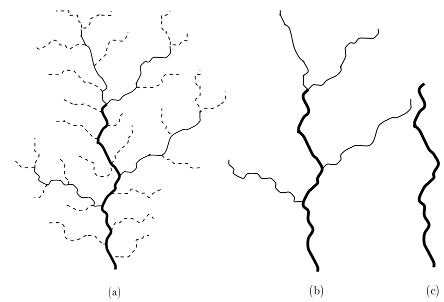




# Properties

## 8. Horton-Strahler stream ordering:

- Metrics for branching networks:
  - Method for ordering streams hierarchically
  - Reveals fractal nature of natural branching networks
  - Hierarchy is not pure but mixed (Tokunaga).[18, 8]
  - Major examples: rivers and blood networks.




- Beautifully described but poorly explained.

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# Properties

## 9. Network distances:

### (a) shortest path length $d_{ij}$:

- Fewest number of steps between nodes $i$ and $j$.
- (Also called the chemical distance between $i$ and $j$.)

### (b) average path length $\langle d_{ij} \rangle$:

- Average shortest path length in whole network.
- Good algorithms exist for calculation.
- Weighted links can be accommodated.

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## 9. Network distances:

### (c) Network diameter $d_{\max}$:

- Maximum shortest path length in network.

### (d) Closeness $d_{\rm cl} = [\sum_{ij} d_{ij}^{-1} / \binom{n}{2}]^{-1}$:

- Average 'distance' between any two nodes.
- Closeness handles disconnected networks ($d_{ij} = \infty$)
- $d_{\rm cl} = \infty$ only when all nodes are isolated.

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## 10. Centrality:

- Many such measures of a node's 'importance.'
- ex 1: Degree centrality: $k_i$.
- ex 2: Node $i$'s betweenness
  = fraction of shortest paths that pass through $i$.
- ex 3: Edge $\ell$'s betweenness
  = fraction of shortest paths that travel along $\ell$.
- ex 4: Recursive centrality: Hubs and Authorities (Jon Kleinberg [12])





# Nutshell:

## Overview Key Points:

- The field of complex networks came into existence in the late 1990s.
- Explosion of papers and interest since 1998/99.
- Hardened up much thinking about complex systems.
- Specific focus on networks that are large-scale, sparse, natural or man-made, evolving and dynamic, and (crucially) measurable.
- Three main (blurred) categories:
  1. Physical (e.g., river networks),
  2. Interactional (e.g., social networks),
  3. Abstract (e.g., thesauri).





# Nutshell:

## Overview Key Points (cont.):

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- Two main areas of focus:
  1. Description: Characterizing very large networks
  2. Explanation: Micro story $\Rightarrow$ Macro features
- Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...
- Still much work to be done, especially with respect to dynamics...





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